Finite Plastic Deformation Due to Crystallographic Slip

G. Y. Chin, R. N. Thurston, and E. A. Nesbitt

A general relationship between the amount of glide shear (due to slip) and the macroscopic shape change has been developed. Since the deformation can be large, finite strain analysis is employed. In this treatment, the shape change is expressed by a deformation gradient matrix, $F \equiv [\partial x_i/\partial X_j]$, *where* X *and* x *refer to the initial and final positions of a particle. This matrix is readily evaluated in terms of the amount of glide shear,* a, *and the direction cosines of the slip-plane normal and the slip direction for a single active slip system. In the case of deformation from slip on several systems, the product* $F(a)$ *of the several deformation gradient matrices is first calculated. Then, by assuming that the final configuration is reached by a long series of small shears of magnitude* a, *occurring more or less alternately in the several slip systems, the final deformation gradient matrix can be obtained. Mathematically, the problem is reduced to finding the limit of* $F(a)^N$ *as* $N \to \infty$ *while* $a \to 0$ *in such a way that the product* $Na = \alpha$, *a finite constant designating the accumulated amount of shear. It turns out that this limit is simply* $e^{\alpha F_1}$, *where* F_1 *is the matrix whose elements are the coefficients of* a *in the* $F(a)$ *matrix. Application is made of the present treatment to a fcc crystal of Permalloy compressed on the (110) plane and contrained to elongate in the [$\bar{1}$12] direction. In addition, it is shown that one may readily obtain from the general analysis the well-known formulas relating elongation, amount of glide shear, and amount of lattice rotation for crystals deforming by single and double slip under tension.*

In problems of crystal plasticity, it is often necessary to relate the amount of glide in the operating slip systems to the macroscopic strain components. This relationship is relatively simple when only small strains are considered.[1-2] In this case the separate strain contributions from several slip systems are additive. Such a procedure is incorrect in the case of large plastic deformation. To the authors' knowledge, no general treatment of the latter problem has appeared in the literature. In specific instances, Mark, Polanyi, and Schmid[3] have derived the relationship between glide and axial elongation during tensile pulling of a single crystal when only one slip system is active. A similar relationship for duplex slip was worked out by v. Göler and Sachs.[4] Taylor and Elam[5] have also studied in detail problems of large plastic deformation. Due to uncertainties as to the slip systems at the time, however, they were mainly concerned with proving that slip occurs on $\{111\}\langle 110\rangle$ systems in fcc metals.

The present general treatment arose out of recent investigations of magnetic anisotropy in cold-worked Fe-Ni alloys[6] as well as some related problems in strength anisotropy of single crystals.[7] Detailed application of the general analysis will be provided in the case of a Permalloy single crystal compressed on the (110) plane and constrained to elongate only in the [$\bar{1}$12] direction. It will be seen that, as expected, the small strain approximation leads to significant errors after moderate straining. It will also be shown that the present general treatment yields the formulas of Mark *et al.* and of v. Göler and Sachs in those specific cases.

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GENERAL CONSIDERATIONS[8]

Consider a homogeneous deformation in which a material point initially at (X_1, X_2, X_3) moves to (x_1, x_2, x_3), both positions being referred to the same set of Cartesian axes. Such a deformation can be specified completely by the deformation gradient matrix, F, of components $\partial x_i/\partial X_j$:

$$F \equiv \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \qquad [1]$$

The displacement components are $u_i = x_i - X_i$, and hence the matrix [1] can be written in terms of the displacement derivatives by the substitutions

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \qquad [2]$$

where δ_{ij} is the Kronecker delta.

In view of Eq. [2], the displacement derivatives and the small strain components

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \qquad [3]$$

can readily be obtained from the matrix F.

Since the nine quantities $\partial x_i/\partial X_j$ specify the deformation completely, all quantities associated with the deformation can be derived from them. For example, the ratio of final volume to initial volume is the determinant of F:

$$\frac{V_{\text{final}}}{V_{\text{initial}}} = \det F \qquad [4]$$

The ratio, $\lambda_{\mathbf{P}}$, of final length to initial length for any material line can be found from

$$\lambda_{\mathbf{P}}^2 = \sum_{i,j,k} \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} P_j P_k \qquad [5]$$

where $\mathbf{P}$ is the unit vector giving the initial direction of the line. The unit vector $\mathbf{p}$ giving the final direction of the line (after the deformation) has the components

$$p_i = \frac{1}{\lambda_{\mathbf{P}}} \sum_j \frac{\partial x_i}{\partial X_j} P_j, \qquad i = 1, 2, 3 \qquad [6]$$

In general, the rotation of a material plane and the change in the perpendicular distance between parallel specimen faces is most conveniently expressed in terms of the quantities $\partial X_i/\partial x_j$, which may be obtained by inversion of the matrix F. The ratio $f_{\mathbf{Q}}$ of initial to final perpendicular distance between material planes of initial unit normal $\mathbf{Q}$ can be found from

$$f_{\mathbf{Q}}^2 = \sum_{i,j,k} \frac{\partial X_i}{\partial x_j}\frac{\partial X_k}{\partial x_j} Q_i Q_k \qquad [7]$$

while the planes acquire the final unit normal $\mathbf{q}$ of components

$$q_j = \frac{1}{f_{\mathbf{Q}}} \sum_i Q_i \frac{\partial X_i}{\partial x_j} \qquad [8]$$

If the initial and final normal can be identified as the same material line, then $f_{\mathbf{Q}} = 1/\lambda_{\mathbf{Q}}$.

An important advantage of specifying a deformation by its deformation gradient matrix is that the matrix for the resultant of two or more successive deformations is the product of the matrices for the individual deformations. For example, consider two successive deformations such that the first moves points initially at (X_1, X_2, X_3) to some intermediate configuration (y_1, y_2, y_3) specified by the deformation gradient matrix

$$F_A = \left[\frac{\partial y_i}{\partial X_j}\right] \qquad [9]$$

while the second deformation, from the intermediate configuration (y_1, y_2, y_3) to the final configuration (x_1, x_2, x_3), is specified by

$$F_B = \left[\frac{\partial x_i}{\partial y_j}\right] \qquad [10]$$

Since

$$\frac{\partial x_i}{\partial X_j} = \sum_{k=1}^{3} \frac{\partial y_k}{\partial X_j}\frac{\partial x_i}{\partial y_k}$$

the deformation gradient matrix for the resultant deformation from (X_1, X_2, X_3) to (x_1, x_2, x_3) is given by the matrix product $F_B F_A$, *i.e.*,

$$\left[\frac{\partial x_i}{\partial X_j}\right] = \left[\sum_{k=1}^{3} \frac{\partial x_i}{\partial y_k}\frac{\partial y_k}{\partial X_j}\right] = F_B F_A \qquad [11]$$

Finally, although we have no occasion to use them in the present paper, it may be noted that the finite strain components in the material (or Lagrangian) description are the elements of $(1/2)(F^T F - I)$ while the corresponding quantities in the spatial (Eulerian) description are the elements of $(1/2)[I - (F^{-1})^T F^{-1}]$. Here I is the unit matrix and the superscript T denotes the transpose.

SINGLE SLIP

Consider a unit vector $\mathbf{m}$ (with components m_1, m_2, m_3) along the slip direction and a unit vector $\mathbf{n}$ (components n_1, n_2, n_3) normal to the slip plane. Then, with the origin considered a fixed point, we have

$$u_i = x_i - X_i = a(\mathbf{X}\cdot\mathbf{n})m_i \qquad i = 1, 2, 3 \qquad [12]$$

where $\mathbf{X}$ is the position vector with components X_i, and a is the amount of simple shear resulting from slip. Upon noting that $\mathbf{X}\cdot\mathbf{n} = \sum_j X_j n_j$, we find, by differentiation of [12] in accordance with [2],

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + a m_i n_j \qquad [13]$$

or

$$F = \begin{bmatrix} 1 + am_1 n_1 & am_1 n_2 & am_1 n_3 \\ am_2 n_1 & 1 + am_2 n_2 & am_2 n_3 \\ am_3 n_1 & am_3 n_2 & 1 + am_3 n_3 \end{bmatrix} \qquad [14]$$

Eq. [14] can be written as

$$F = I + amn^T \qquad [14a]$$

where m and n are the single-column matrices of direction cosines, and the superscript T denotes the transpose as before; *i.e.*, n^T is a single-*row* matrix. Since the slip direction $\mathbf{m}$ lies in the slip plane, *i.e.*, perpendicular to $\mathbf{n}$, we always have $\mathbf{m}\cdot\mathbf{n} = 0$. It follows that $\det F = 1$, *i.e.*, there is no volume change, and that F^{-1} for use in the general equations [7] and [8] is simply

$$F^{-1} = I - amn^T \qquad [14b]$$

or

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - am_i n_j \qquad [14c]$$

If we take a Cartesian coordinate system with the normal **n** to the slip plane as axis 1 and the slip direction **m** as axis 2, the components of **n** are (1, 0, 0) and those of **m** are (0, 1, 0). Matrix [14] then becomes

$$F = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [15]$$

With the deformation gradient matrix given by Eq. [15], Eq. [5] reduces to

$$\lambda_{\mathbf{P}}^2 = \left(\frac{l_1}{l_0}\right)^2 = 1 + a^2 P_1^2 + 2aP_1P_2 \qquad [16]$$

while Eq. [6] yields

$$p_1 = \frac{l_0}{l_1} P_1 \qquad [17a]$$

$$p_2 = \frac{l_0}{l_1}(aP_1 + P_2) \qquad [17b]$$

$$p_3 = \frac{l_0}{l_1} P_3 \qquad [17c]$$

Here P_1, P_2, and P_3 are the direction cosines (with respect to the same coordinates as **m** and **n** above) of the initial direction of any arbitrary material line; p_1, p_2, and p_3 are the corresponding values after the deformation; l_1/l_0 is the ratio of final to initial length.

These formulas are applicable to tensile testing when the deformation corresponds to a single active slip system. The grip system maintains the direction of the material line along the tensile axis. This line, however, rotates with respect to the lattice, and hence with respect to our coordinate system which is fixed in the lattice. With **P** along the tensile axis, the above formulas enable one to find the length ratio l_1/l_0 and the rotation of the tensile axis with respect to the lattice. The amount of shear, a, can be expressed in terms of the initial and final positions of the tensile axis by solving Eq. [17b] for a after substituting for l_0/l_1 from [17a]. The result is

$$a = \frac{p_2}{p_1} - \frac{P_2}{P_1} \qquad [18]$$

Eqs. [16] to [18] have been derived previously by Mark, Polanyi, and Schmid.[3]

As a specific application, Fig. 1 shows a standard (001) stereographic projection. If the tensile axis P of a single-crystal rod lies anywhere within the standard [001]–[$\bar{1}$11]–[011] triangle, then according to the Schmid law the active slip system for a fcc crystal is (111)[$\bar{1}$01] (the primary slip system). It is convenient to use [111], [$\bar{1}$01], [$1\bar{2}1$] as Cartesian coordinate axes, in which case the deformation gradient matrix is given by Eq. [15], and the remaining formulas [16] to [18] are directly applicable.

TWO OR MORE SLIP SYSTEMS

In extending the treatment to two (or more) slip systems A and B, we express the corresponding deformation gradient matrices (see Eq. [14a]) as

$$F_A = I + am_A n_A^T$$
$$F_B = I + bm_B n_B^T = I + \beta a m_B n_B^T \qquad [19]$$

where $\beta = b/a$ is the ratio of glide-shear of the two slip systems. If shear in A is followed by shear in B, the deformation gradient matrix for the combination is

$$F_BF_A = I + a(m_A n_A^T + \beta m_B n_B^T) + a^2\beta(m_B n_B^T m_A n_A^T)$$
$$= I + aF_1 + a^2 F_2 \qquad [20]$$

where $F_1 = m_A n_A^T + \beta m_B n_B^T$ and $F_2 = \beta m_B n_B^T m_A n_A^T$. If shear in B is followed by shear in A, the combined result, given by F_AF_B, is the same except that $F_2 = \beta m_A n_A^T m_B n_B^T$. Note in general $m_A n_A^T m_B n_B^T \neq m_B n_B^T m_A n_A^T$ since this is a matrix product.

Physically, we imagine that the final configuration resulting from the operation of the two slip systems is reached by a long series of steps in which a *small* deformation F_A (or F_B) is followed by a *small* deformation F_B (or F_A). Thus, we expect to represent the final configuration mathematically by a deformation gradient matrix which is the limit of $(F_BF_A)^N$ as $N \to \infty$ while $a \to 0$ in such a way that the product $Na = \alpha$, a finite constant designating the accumulated amount of shear in slip system A. The desired limit is

$$F = \lim_{a \to 0} (F_BF_A)^{\alpha/a} = \lim_{a \to 0} (I + aF_1 + a^2F_2)^{\alpha/a}$$
$$= I + \alpha F_1 + \frac{1}{2}\alpha^2 F_1^2 + \frac{1}{3!}\alpha^3 F_1^3 + \cdots$$
$$= e^{\alpha F_1} \qquad [21]$$

Since F_2 does not enter the final result, $(F_AF_B)^N$ has the same limit. Thus, as expected, the final configuration is independent of the exact sequence of opera-

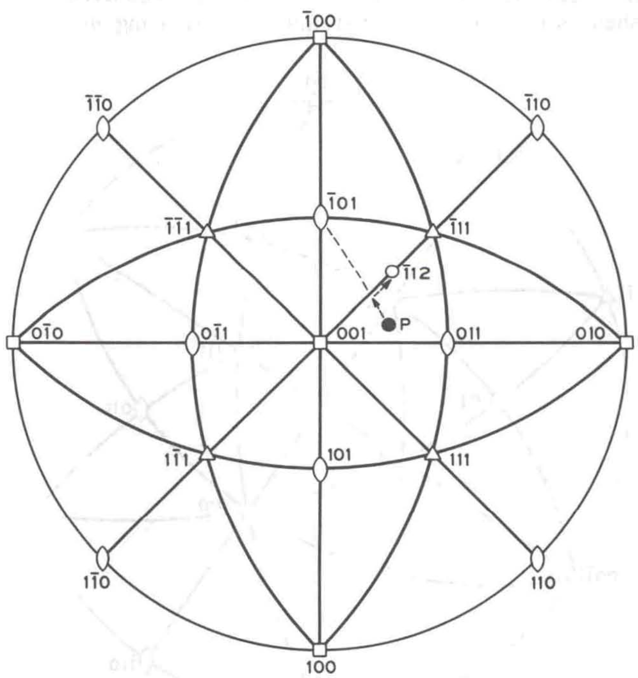

Fig. 1—Standard (001) stereographic projection for a cubic crystal. A single crystal whose tensile axis P lies inside the triangle deforms by (111)[$\bar{1}$01] (primary) slip. If tensile axis lies along the [001]-[$\bar{1}$11] line, equal slip on (111)[$\bar{1}$01] (primary) and ($\bar{1}\bar{1}$1)[011] (conjugate) results. Arrows indicate path of axial rotation.

tion of the two slip systems, in contrast to the result of a sequence of two *finite* shears. It can easily be shown that Eq. [21] is likewise applicable to more than two slip systems. The matrix F_1 will in general take the form $F_1 = m_A n_A^T + \beta m_B n_B^T + \gamma m_C n_C^T + \cdots$.

It remains now to evaluate the matrix $e^{\alpha F_1}$ of Eq. [21]. We first note that any similarity transformation that diagonalizes αF_1 also diagonalizes $e^{\alpha F_1}$. Let us suppose that a nonsingular matrix S has been found such that

$$S(\alpha F_1)S^{-1} = \alpha \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \equiv \alpha\Lambda \qquad [22]$$

Then,

$$e^{\alpha\Lambda} = S(e^{\alpha F_1})S^{-1} = \begin{bmatrix} e^{\alpha\lambda_1} & 0 & 0 \\ 0 & e^{\alpha\lambda_2} & 0 \\ 0 & 0 & e^{\alpha\lambda_3} \end{bmatrix} \qquad [23]$$

and $e^{\alpha F_1}$ can be found from

$$e^{\alpha F_1} = S^{-1}[S(e^{\alpha F_1})S^{-1}]S = S^{-1}(e^{\alpha\Lambda})S \qquad [24]$$

This evaluation requires diagonalization of the matrix. E. N. Gilbert[9] has shown us an elegant method, presented in the Appendix, of evaluating $e^{\alpha F_1}$ without diagonalization. The calculations will now be illustrated with the following deformation.

CASE OF (110)[1̄12] COMPRESSION

Fig. 2 shows the standard (110) stereographic projection. If a fcc single crystal is compressed on the (110) plane and constrained to flow in the [1̄12] direction (by confining the crystal to a channel), slip will occur equally in the two systems $A \equiv (111)[10\bar{1}]$ and $B \equiv (11\bar{1})[011]$ as a result of a favorable resolved shear stress on these systems. In evaluating m_A, m_B,

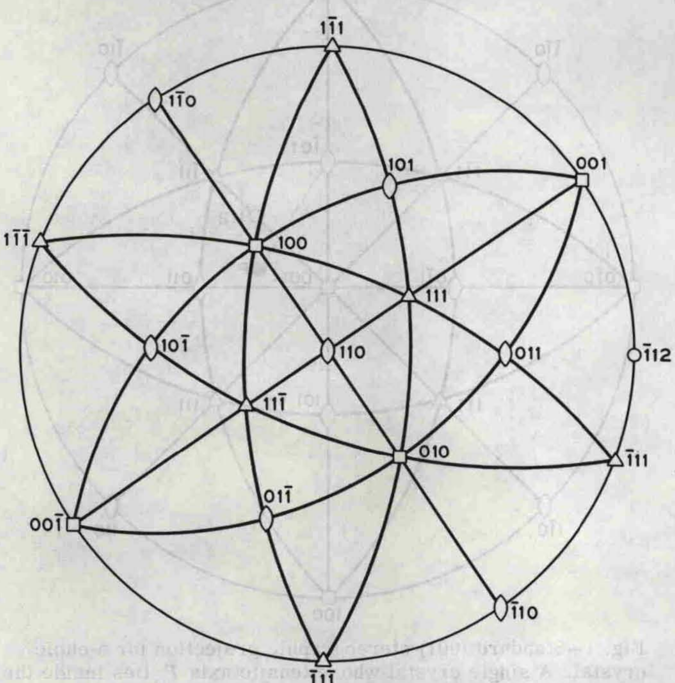

Fig. 2—Standard (110) stereographic projection. For compression on (110) and elongation in [1̄12], the active slip systems are (111̄)[011] and (111)[101̄].

and so forth, of Eq. [20], it is convenient to take the specimen axes as Cartesian coordinates, *i.e.*, let X_1, X_2, X_3 be respectively along [110], [1̄11̄], [1̄12], Fig. 2. The matrix of transformation from cubic axes to those above is

	[100]	[010]	[001]
[110]	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
[1̄11̄]	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
[1̄12]	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$

[25]

Hence, if $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are unit vectors along the specimen axes and $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ along the cubic axes, we have, for (111)[101̄] slip,

$$\mathbf{n}_A = \frac{1}{\sqrt{3}}(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3) = \frac{2}{\sqrt{6}}\mathbf{i}_1 - \frac{1}{3}\mathbf{i}_2 + \frac{\sqrt{2}}{3}\mathbf{i}_3$$

$$\mathbf{m}_A = \frac{1}{\sqrt{2}}(\mathbf{I}_1 - \mathbf{I}_3) = \frac{1}{2}\mathbf{i}_1 - \frac{\sqrt{3}}{2}\mathbf{i}_3 \qquad [26]$$

and for (111̄)[011] slip,

$$\mathbf{n}_B = \frac{2}{\sqrt{3}}(\mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_3) = \frac{2}{\sqrt{6}}\mathbf{i}_1 + \frac{1}{3}\mathbf{i}_2 - \frac{\sqrt{2}}{3}\mathbf{i}_3$$

$$\mathbf{m}_B = \frac{1}{\sqrt{2}}(\mathbf{I}_2 + \mathbf{I}_3) = \frac{1}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_3 \qquad [27]$$

From Eqs. [14] and [26], we have

$$F_A = \begin{bmatrix} 1 - \frac{a}{\sqrt{6}} & \frac{a}{6} & -\frac{a}{3\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{a}{\sqrt{2}} & -\frac{a}{2\sqrt{3}} & 1 + \frac{a}{\sqrt{6}} \end{bmatrix} \qquad [28]$$

where shear in the negative sense has been chosen to conform with compression along X_1.

Similarly, Eqs. [14] and [27] yield

$$F_B = \begin{bmatrix} 1 - \frac{b}{\sqrt{6}} & -\frac{b}{6} & \frac{b}{3\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{b}{\sqrt{2}} & -\frac{b}{2\sqrt{3}} & 1 + \frac{b}{\sqrt{6}} \end{bmatrix} \qquad [29]$$

Hence in the expected case of equal slip, $b = a$,

$$F_B F_A = \begin{bmatrix} 1 - \frac{2a}{\sqrt{6}} + \frac{a^2}{3} & -\frac{a^2}{3\sqrt{6}} & \frac{a^2}{3\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{a^2}{\sqrt{3}} & -\frac{a}{\sqrt{3}} - \frac{a^2}{3\sqrt{2}} & 1 + \frac{2a}{\sqrt{6}} + \frac{a^2}{3} \end{bmatrix} \qquad [30]$$

In the form of Eq. [20], $F_B F_A = I + aF_1 + a^2 F_2$,

$$F_1 = \begin{bmatrix} -\frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$F_2 = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3\sqrt{6}} & \frac{1}{3\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix} \qquad [31]$$

In evaluating the final deformation gradient matrix $e^{\alpha F_1}$ (Eq. [21]) by the use of Eqs. [22] to [24], the procedure is straightforward and can be found in standard texts. The elements of the diagonal matrix Λ (Eq. [22]) are found to be $\lambda_1 = -2/\sqrt{6}$, $\lambda_2 = 0$, $\lambda_3 = 2/\sqrt{6}$, and suitable matrices for the similarity transformation of Eqs. [22] and [23] are

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \qquad [32]$$

The application of Eqs. [23] and [24] then leads to the desired deformation gradient matrix

$$F = e^{\alpha F_1} = S^{-1} \begin{bmatrix} e^{-\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{\sqrt{2}} e^{\varphi} & e^{\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1 - e^{\varphi}) & e^{\varphi} \end{bmatrix} \qquad [33]$$

where we have substituted $\varphi \equiv 2\alpha/\sqrt{6}$ merely as an abbreviation.

To obtain the formulas relating the changes in specimen dimensions and angles of lines, it may be shown that application of Eqs. [5] to [7] and [33] leads to values of

$$\frac{h_1}{h_0} = e^{-\varphi}$$

$$\frac{w_1}{w_0} = 1$$

$$\frac{l_1}{l_0} = e^{\varphi} \qquad [34]$$

$$\tan \theta_{32} = \frac{1}{\sqrt{2}}(1 - e^{\varphi})$$

where h_1, w_1, l_1 and h_0, w_0, l_0 refer to specimen height, width, and length after and before the deformation, respectively, and θ_{32} is the angle by which a line originally parallel to the $[1\bar{1}\bar{1}]$ direction has shifted in the $[\bar{1}12]$ direction.

By contrast, it can be shown that the use of infinitesimal strain equations [3] leads to the following expressions:

$$\frac{h_1}{h_0} = 1 - \varphi$$

$$\frac{w_1}{w_0} = 1$$

$$\frac{l_1}{l_0} = 1 + \varphi$$

$$\tan \theta_{32} = -\frac{\sqrt{2}}{2}\varphi \qquad [35]$$

which may also be deduced from Eq. [34] for *small* values of φ. Clearly for large values of φ, Eq. [35] can be quite erroneous.

Experiment. A single crystal of Permalloy (4 pct Mo-17 pct Fe-79 pct Ni) was compressed on the (110) plane with the elongation confined to the $[\bar{1}12]$ direction. A special compression die was constructed to restrict lateral spreading of the specimen. As illustrated in Fig. 3, the die consists of a slot formed by three steel blocks bolted together to facilitate specimen removal. After the sample is placed in the slot a plunger is fitted on top and the ensemble placed in a Baldwin hydraulic machine for compression testing. Good lubrication was achieved with 5-mil-thick Teflon strips. Periodically the sample was removed for dimensional measurement as well as for renewal of the Teflon. A more detailed description of the experiment is given in Ref. 7.

The specimen shape after a 50.5 pct reduction in height is shown in Fig. 4. Metallographic observations of the slip-line tracings indicate that slip had occurred primarily on the two expected systems $(111)[10\bar{1}]$ and $(11\bar{1})[011]$, Fig. 5. From Eq. [34] one obtains

$$\tan \theta_{32} = \frac{1}{\sqrt{2}}\left(1 - \frac{h_0}{h_1}\right) \qquad [36]$$

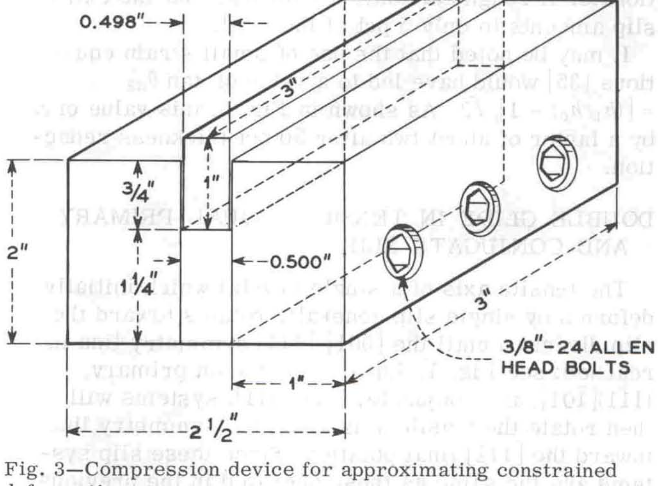

Fig. 3—Compression device for approximating constrained deformation.

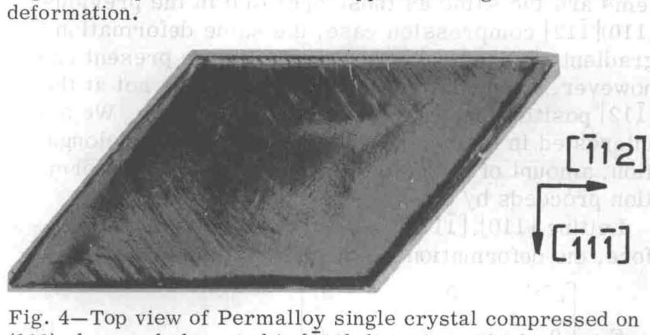

Fig. 4—Top view of Permalloy single crystal compressed on (110) plane and elongated in $[\bar{1}12]$ direction. Thickness reduction 50.5 pct. Initial rectangular shape has changed to a parallelogram. Directions noted in margins. X2.70. Reduced approximately 1 pct for reproduction.

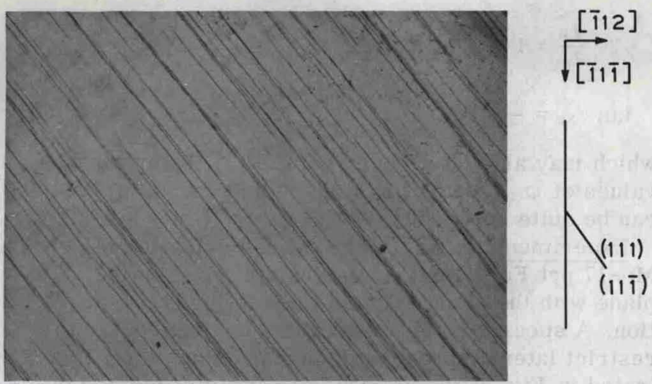

Fig. 5—Slip traces on top surface of (110)[1̄12] Permalloy crystal. Compressed 50.5 pct, electropolished, and then lightly compressed. Traces correspond to both (111) and (111̄) slip planes. X140. Reduced approximately 34 pct for reproduction.

and hence a plot of $\tan\theta_{32}$ vs $(1 - h_0/h_1)$ should yield a straight line with a slope of 0.707. This is shown in Fig. 6, together with the experimentally determined points. Agreement is considered very good. The slight positive deviation from the expected line can be explained on the basis of a small activity on the systems (111)[011̄] and (111̄)[101]; see Fig. 2. If lateral constraint had been absent, all four slip systems would have been equally favored. It can be shown that slip on (111)[011̄] and (111̄)[101] systems will cause lateral spreading as well as contributing to a larger value of θ_{32}. The measured width of the sample was found to increase from 0.498 to 0.509 in. after 50.5 pct reduction, supporting the above rationale. A rough calculation indicates that the extra slip amounts to only 5 pct of the total.

It may be noted that the use of small strain equations [35] would have led to a value of $\tan\theta_{32} = [(h_1/h_0) - 1]/\sqrt{2}$. As shown in Fig. 6, this value errs by a factor of about two after 50 pct thickness reduction.

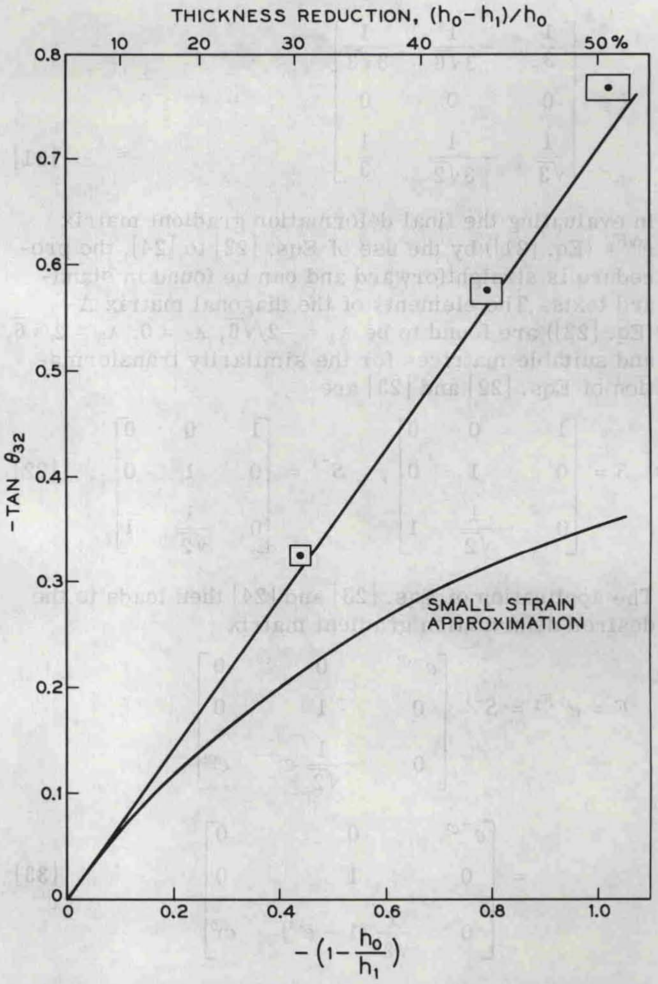

Fig. 6—Plot of $\tan\theta_{32}$ vs $(1 - h_1/h_0)$ for (110)[1̄12] compression. Straight line is according to theory, assuming only (111̄)[011] and (111)[101̄] slip. Measured points from experiment on Permalloy single crystal. Size of square containing measured point indicates range of errors in measurement.

DOUBLE GLIDE IN TENSION—EQUAL PRIMARY AND CONJUGATE SLIP

The tensile axis of a single crystal which initially deforms by single slip generally rotates toward the slip direction until the [001]-[1̄11] symmetry line is reached; see Fig. 1. Equal slip on both primary, (111)[1̄01], and conjugate, (1̄1̄1)[011], systems will then rotate the tensile axis along the symmetry line toward the [1̄12] final position. Since these slip systems are the same as those operated in the previous (110)[1̄12] compression case, the same deformation gradient matrix [33] may be used. In the present case, however, the initial elongation direction is not at the [1̄12] position, but instead at ϵ_0 from it, say. We are interested in the relationship between tensile elongation, amount of glide, and lattice rotation as deformation proceeds by double glide.

Letting [110], [1̄11̄], [1̄12] be coordinate axes as before, the deformation gradient matrix from [33] is

$$F = \begin{bmatrix} e^{-\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}\,(1 - e^{\varphi}) & e^{\varphi} \end{bmatrix} \quad [37]$$

For a tensile axis initially at ϵ_0 degrees from [1̄12] and toward [001], the unit vector $\mathbf{P}$ along this axis is

$$\mathbf{P} = -\sin\epsilon_0\mathbf{i}_2 + \cos\epsilon_0\mathbf{i}_3 \quad [38]$$

or

$$P_1 = 0, \quad P_2 = -\sin\epsilon_0, \quad P_3 = \cos\epsilon_0$$

Let the tensile axis move to a position ϵ_1 deg from [1̄12] after the deformation with the line $\mathbf{p}$ along this axis being

$$\mathbf{p} = -\sin\epsilon_1\mathbf{i}_2 + \cos\epsilon_1\mathbf{i}_3 \quad [39]$$

or

$$p_1 = 0, \quad p_2 = -\sin\epsilon_1, \quad p_3 = \cos\epsilon_1$$

From Eq. [6], which relates the direction cosines of the two lines, we have

$$\lambda_{\mathbf{p}} p_2 = \frac{\partial x_2}{\partial X_1}P_1 + \frac{\partial x_2}{\partial X_2}P_2 + \frac{\partial x_2}{\partial X_3}P_3$$

or

$$-\lambda_{\mathbf{p}}\sin\epsilon_1 = -\sin\epsilon_0$$

or

$$\lambda_P = \frac{l_1}{l_0} = \frac{\sin \epsilon_0}{\sin \epsilon_1} \qquad [40]$$

Eq. [40], which relates the tensile elongation with the lattice rotation for a crystal undergoing double glide, was obtained by v. Göler and Sachs[4] through the integration of a differential equation. From Eq. [6], we also have

$$\lambda_P p_3 = \frac{\partial x_3}{\partial X_1} P_1 + \frac{\partial x_3}{\partial X_2} P_2 + \frac{\partial x_3}{\partial X_3} P_3$$

or

$$\lambda_P \cos \epsilon_1 = -\frac{1}{\sqrt{2}} (1 - e^{\varphi}) \sin \epsilon_0 + e^{\varphi} \cos \epsilon_0$$

or

$$e^{\varphi} = \frac{\sqrt{2} \cot \epsilon_1 + 1}{\sqrt{2} \cot \epsilon_0 + 1} \qquad [41]$$

by substituting $\lambda_P = \sin \epsilon_0 / \sin \epsilon_1$. Eq. [41] may be rewritten as

$$S = 2\alpha = \sqrt{6}\varphi = \sqrt{6} \ln \left[\frac{\sqrt{2} \cot \epsilon_1 + 1}{\sqrt{2} \cot \epsilon_0 + 1}\right] \qquad [42]$$

which relates the amount of glide and the lattice rotation for the double-glide case. Eq. [42] was likewise developed by v. Göler and Sachs. It may be noted if ϵ_0 and ϵ_1 are measured from $[\bar{1}12]$ and toward the $[\bar{1}11]$ position, Eq. [42] becomes

$$S = \sqrt{6} \ln \left[\frac{\sqrt{2} \cot \epsilon_1 - 1}{\sqrt{2} \cot \epsilon_0 - 1}\right] \qquad [43]$$

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*Note added in proof: After the present work was submitted for publication, two related papers, by Bowen and Christian[10] and by Schubert,[11] have come to our attention. The Bowen and Christian treatment of single glide is essentially the same as ours. Their results for double glide, like those of v. Göler and Sachs, were obtained by integrating a differential equation. The latter method was also used by Schubert in treating both single and double glide. On the other hand, we obtained, directly from the limit in Eq.[21], the resultant deformation gradient matrix, from which all quantities associated with the deformation can be computed readily.

APPENDIX[9]

ALTERNATIVE EVALUATION OF $e^{\alpha F_1}$ IN THE DOUBLE-GLIDE CASE

In Eq. [20], it is noted that

$$F_1 = m_A n_A^T + \beta m_B n_B^T \qquad [A.1]$$

for slip on systems A and B. Let us define matrices

$$P = m_A n_A^T, \quad Q = m_B n_B^T, \quad R = m_B n_A^T, \quad S = m_A n_B^T \qquad [A.2]$$

and the scalar products

$$r = n_A^T m_B, \qquad s = n_B^T m_A \qquad [A.3]$$

Since the slip directions $\mathbf{m}_A$ and $\mathbf{m}_B$ lie in the slip planes of normals $\mathbf{n}_A$ and $\mathbf{n}_B$, respectively, we have

$$n_A^T m_A = n_B^T m_B = 0 \qquad [A.4]$$

In view of the above definitions, the matrices P, Q, R, and S have the following multiplication table:

Second Factor

First Factor		P	Q	R	S
	P	0	rS	rP	0
	Q	sR	0	0	sQ
	R	0	rQ	rR	0
	S	sP	0	0	sS

[A.5]

By application of this table, one finds

$$F_1^2 = (P + \beta Q)^2 = P^2 + \beta(QP + PQ + \beta Q^2) = \beta(sR + rS)$$

$$F_1^3 = (P + \beta Q)\beta(sR + rS) = \beta(sPR + rPS) + \beta^2(sQR + rQS) = \beta rs(P + \beta Q) = \beta rs F_1$$

It can be seen that each even power of F_1 is a scalar multiple of the matrix F_1^2, which we denote by

$$B = F_1^2 = \beta(sR + rS) \qquad [A.6]$$

while each odd power is a scalar multiple of F_1 itself, since

$$BF_1 = F_1 B = \beta rs F_1 \qquad [A.7]$$

Thus

$$F_1^2 = B, \qquad F_1^3 = \beta rs F_1$$
$$F_1^4 = B^2 = \beta rs B, \quad F_1^5 = (\beta rs)^2 F_1, \text{ and so forth} \qquad [A.8]$$

In general,

$$F_1^{2k+1} = (\beta rs)^k F_1$$
$$F_1^{2k+2} = (\beta rs)^k B \qquad [A.9]$$

Finally,

$$e^{\alpha F_1} = I + \sum_{k=0}^{\infty} \frac{(\alpha F_1)^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(\alpha F_1)^{2k+2}}{(2k+2)!}$$

$$= I + \frac{F_1}{\sqrt{\beta rs}} \sum_{k=0}^{\infty} \frac{(\alpha\sqrt{\beta rs})^{2k+1}}{(2k+1)!} + \frac{B}{\beta rs} \sum_{k=0}^{\infty} \frac{(\alpha\sqrt{\beta rs})^{2k+2}}{(2k+2)!}$$

$$= I + \frac{F_1}{\sqrt{\beta rs}} \sinh(\alpha\sqrt{\beta rs}) + \frac{F_1^2}{\beta rs} [\cosh(\alpha\sqrt{\beta rs}) - 1] \qquad [A.10]$$

As a simple example, we reconsider the case of $(110)[\bar{1}12]$ compression, for which $e^{\alpha F_1}$ has already been evaluated in Eq. [33]. For this case, we have $\beta = 1$, $r = s = (1/3)\sqrt{6} = 2/\sqrt{6}$, and F_1 is given by Eq. [31]. Hence

$$\frac{F_1}{\sqrt{\beta rs}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}, \quad \frac{F_1^2}{\beta rs} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

Substitution into Eq. [A.10] then gives, with $\varphi = 2a/\sqrt{6}$,

$$e^{\alpha F_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \sinh \varphi + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} (\cosh \varphi - 1)$$

$$= \begin{bmatrix} (\cosh \varphi - \sinh \varphi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} (\cosh \varphi + \sinh \varphi - 1) & (\cosh \varphi + \sinh \varphi) \end{bmatrix} = \begin{bmatrix} e^{-\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} (1 - e^{\varphi}) & e^{\varphi} \end{bmatrix}$$

in agreement with Eq. [33].

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